MATH 130A Review: Euler Number and Limits

Facts to Know

- Euler's number

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$$e \approx 2.718\ldots$$

$$f(x) = \underline{b^x}$$

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \underline{b^x} \cdot \underbrace{\lim_{h\to 0} \frac{b^h - 1}{h}}_{=1}$$

$$\frac{d}{dx} e^x = e^x$$

- Taylor series expansion of a function $f(x)$ centered at zero

$$f(x) = \sum_{n=0}^{\infty} c_n \cdot x^n$$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

If $f(x) = e^x$

then $f^{(n)}(x) = e^x$

$$c_n = \frac{e^0}{n!} = \frac{1}{n!}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n}$$

- Miscellaneous facts
  - $\boxed{\lim_{n\to\infty} (1 + \frac{1}{n})^n = e}$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \equiv \lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} = 1$$

  - $\boxed{\lim_{n\to\infty} (1 + \frac{\lambda}{n})^n = e^\lambda}$

$$\lim_{n\to\infty} \begin{cases} n(e^{1/n} - 1) = 1 \\ e^{1/n} - 1 = \frac{1}{n} \\ e^{1/n} = 1 + \frac{1}{n} \\ e = (1 + \frac{1}{n})^n \end{cases}$$

  - $\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = e}$

$x = 1$

  - $\boxed{\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^\lambda}$

Examples

- Fill in the table with a calculator the value of $(1 + \frac{\lambda}{n})^n$

	$n = 100$	$n = 1000$	$n = 10,000$	$n = \infty$
$\lambda = 0.05$				$e^{0.05}$
$\lambda = 0.10$				$e^{0.10}$
$\lambda = 0.15$				$e^{0.15}$
$\lambda = 1$				$e^1 = e$ $= 2.718\ldots$

$$\left(1 + \frac{\lambda}{n}\right)^n \approx$$

- Determine the exact limit value of the expression (Hint: write the expression using Σ-notation)

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + \cdots$$

alternating sign

$$\sum_{n=0}^{\infty} \frac{1}{n!} \cdot (-1)^n = e^{(-1)} = \frac{1}{e}$$

- Determine the exact limit value of the expression

$$\lim_{n\to\infty} (1 - \frac{\pi}{n})^n = \left(1 + \frac{(-\pi)}{n}\right)^n \underset{n\to\infty}{=} e^{-\pi}$$

$$\lim_{n\to\infty} \left(1 + \frac{\lambda}{n}\right)^n = e^{\lambda}$$